Communications in Algebra®, 43: 384–399, 2015

ISSN: 0092-7872 print/1532-4125 online
DOI: 10.1080/00927872.2013.842246

Taylor & Francis
Taylor & Francis Group

ABSOLUTELY *s*-PURE MODULES AND NEAT-FLAT MODULES

Engin Büyükaşık and Yılmaz Durğun
Department of Mathematics, Izmir Institute of Technology, Izmir, Turkey

Let R be a ring with an identity element. We prove that R is right Kasch if and only if injective hull of every simple right R-modules is neat-flat if and only if every absolutely pure right R-module is neat-flat. A commutative ring R is hereditary and noetherian if and only if every absolutely s-pure R-module is injective and R is nonsingular. If every simple right R-module is finitely presented, then (1) $_RR$ is absolutely s-pure if and only if R is right Kasch and (2) R is a right $\sum$-CS ring if and only if every pure injective neat-flat right R-module is projective if and only if every absolutely s-pure left R-module is injective and R is right perfect. We also study enveloping and covering properties of absolutely s-pure and neat-flat modules. The rings over which every simple module has an injective cover are characterized.

Key Words: Absolutely *s*-pure module; Injective cover; Kasch ring; Neat submodule; Simple-projective module; *s*-Pure submodule.

2010 Mathematics Subject Classification: 16D10; 16D40; 16D60; 16E30; 16L60.

1. INTRODUCTION

Throughout, R is a ring with an identity element and all modules are unital R-modules. $\mathcal{M}od$-R denotes the category of right R-modules and M_R denotes a right R-module. For an R-module M, the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is denoted by M^+, the dual module $\mathrm{Hom}_R(M, R)$ is denoted by M^*, and $\delta_M : M \to M^{**}$ stands for the evaluation map. M is said to be *torsionless* if δ_M is a monomorphism. We use the notation $E(M)$, $\mathrm{Soc}(M)$, $Rad(M)$, $Z(M)$ for the injective hull, socle, radical, singular submodule of M respectively. Also $J(R)$ denotes the Jacobson radical of a ring R. By $N \leq M$, we mean that N is a submodule of M.

Recently, there is a significant interest to some classes of modules that are defined via simple modules, (see, [1], [4], [5], [6], [9], [20], [21], [25], [37], [38]).

Let $\mathbb{E} : 0 \to A \stackrel{f}{\to} B \to C \to 0$ be a short exact sequence of right R-modules. Following [4], $\mathbb{E}$ is said to be *s-pure* if $f \otimes 1_S : A \otimes S \to B \otimes S$ is a monomorphism for every simple left R-module S. If f is the inclusion homomorphism and $\mathbb{E}$ is *s*-pure, then A is said to be an *s*-pure submodule of B. Similar to the well-known notion of absolutely pure (or FP-injective) module, a right R-module M is called

Received June 13, 2013. Communicated by E. Puczylowski.
Address correspondence to Dr. Yılmaz Durğun, Department of Mathematics, Izmir Institute of Technology, Gülbahçeköyü, 35430, Urla, Izmir, Turkey; E-mail: yilmazdurgun@iyte.edu.tr

absolutely s-pure if it is s-pure in every right R-module N that contains M as a submodule.

Another class of modules related to simple modules are the simple-projective modules. Namely, a right R-module M is said to be *simple-projective* if for any simple right R-module S, every homomorphism $S \to M$ factors through a finitely generated free right R-module. Simple-projective modules introduced in [21] in order to characterize the rings whose simple modules have a projective (pre)envelope. Simple-projective modules and a generalization of these modules have been also studied by Parra and Rada in [29].

A submodule N of a right R-module M is said to be *neat in* M if for any simple right R-module S, every homomorphism $f: S \to M/N$ can be lifted to a homomorphism $g: S \to M$. Neat submodules recently studied in [12], where Fuchs characterizes the integral domains over which s-pure and neat submodules coincide. A right R-module N is said to be *neat-flat* if for any right R-module M and epimorphism $f: M \to N$, the induced map $f^*: \mathrm{Hom}(S, M) \to \mathrm{Hom}(S, N)$ is surjective for any simple right R-module S, that is, every short exact sequence of the form $0 \to K \xrightarrow{f} M \to N \to 0$ is neat-exact, i.e., $f(K)$ is a neat submodule of M.

Neat-flat modules are studied in [1], where it is shown that the notions of simple-projective and neat-flat modules coincide. One of the importance of absolutely s-pure and neat-flat modules is the fact that they are homological objects of some certain proper classes (in the sense of [8]) induced by simple R-modules, (see, Lemma 3.3 and [1, Lemma 3.1]). From another point of view, absolutely s-pure and neat-flat modules, are similar to that of injective and flat modules, respectively. In this regard, it is of interest to investigate the connection between these modules and the rings that are characterized via absolutely s-pure and neat-flat modules. The rings whose simple right modules have a projective preenvelope are characterized by using simple-projective modules (see [20]). At this point, it is natural to consider the rings whose simple right R-modules have an injective cover. These constitute the main objective of the article.

The scheme of the paper is as follows. The properties of absolutely s-pure and neat-flat modules are studied. Some connections between these modules are established. For a right N-ring, i.e., the rings whose simple right modules are finitely presented, we prove that a left R-module M is absolutely s-pure if and only if $\mathrm{Ext}^1_R(Tr(S), M) = 0$ for each simple right R-module S; a right R-module M is neat-flat if and only if M^+ is absolutely s-pure. For a commutative ring, we prove that every absolutely s-pure left R-module is injective and R is nonsingular if and only if R is hereditary and Noetherian. In particular, a domain R is Dedekind if and only if every absolutely s-pure R-module is injective. A ring R is right Kasch if and only if the injective hull of every simple right R-module is neat-flat if and only if for every free left R-module F, F^+ is neat-flat. For a right N-ring, we show that, ${}_RR$ is absolutely s-pure if and only if R is right Kasch; R is right $\sum$-CS if and only if every pure injective neat-flat right R-module is projective if and only if every absolutely s-pure left R-module is injective and R is right perfect.

The last section is devoted for the study of enveloping and covering properties of absolutely s-pure and neat-flat modules. For a right N-ring R, we show that every quotient of any injective left R-module is absolutely s-pure if and only if every left R-module has a monic absolutely s-pure cover if and only if R is a right PS ring;

R is a right Kasch ring if and only if every left R-module has an epic absolutely s-pure cover. For a commutative ring R, we show that, every simple R-module has an injective cover if R is a coherent ring; every simple R-module has a monic injective cover if and only if every R-module has a monic absolutely s-pure cover if and only if a simple R-module S is either injective or $\mathrm{Hom}(E, S) = 0$ for every injective R-module E.

2. PRELIMINARIES

In this section, we give some known results and terminology which are needed to present our results.

Let M be a finitely presented right R-module, that is, M has a free presentation $F_1 \to F_0 \to M \to 0$ where F_0 and F_1 are finitely generated free modules. If we apply the functor $\mathrm{Hom}_R(., R)$ to this presentation, we obtain the sequence

$$0 \to M^* \to F_0^* \to F_1^* \to Tr(M) \to 0,$$

where $Tr(M)$ is the cokernel of the dual map $F_0^* \to F_1^*$. Note that, $Tr(M)$ is a finitely presented left R-module. The left R-module $Tr(M)$ is called an *Auslander–Bridger transpose* of the right R-module M.

In [12], a commutative domain R is called an N-domain if every maximal ideal of R is projective (finitely generated). A ring R is called a *right N-ring* if every maximal right ideal of R is finitely generated. Over a right N-ring every simple right R-module S and its transpose $Tr(S)$ are finitely presented.

The following theorem have a crucial role while obtaining some relations between absolutely s-pure and neat-flat modules.

Theorem 2.1 ([34, Theorem 8.3]). *Let $\mathcal{M}$ be a set of finitely presented right R-modules. For any short exact sequence $\mathbb{E}$ of left R-modules and $M \in \mathcal{M}$, the sequence* $\mathrm{Hom}(Tr(M), \mathbb{E})$ *is exact if and only if the sequence $M \otimes \mathbb{E}$ is exact.*

Let $\mathcal{M}$ be a class of R-modules and M be an R-module. Following [10], we say that a homomorphism $\phi : A \to M$ is an $\mathcal{M}$-precover of M if $A \in \mathcal{M}$ and the abelian group homomorphism $\mathrm{Hom}(A', \phi) : \mathrm{Hom}(A', A) \to \mathrm{Hom}(A', M)$ is surjective for each $A' \in \mathcal{M}$. An $\mathcal{M}$-precover $\phi : A \to M$ is said to be an $\mathcal{M}$-cover of M if every endomorphism $g : A \to A$ such that $\phi g = \phi$ is an isomorphism. $\mathcal{M}$-preenvelope and $\mathcal{M}$-envelope are defined dually. $\mathcal{M}$-covers (respectively, $\mathcal{M}$-envelopes) may not exist in general, but if they exist, they are unique up to isomorphism (see, [39, Proposition 1.2.1]).

The following result is useful while proving whether a class of modules is preenveloping or covering.

Lemma 2.2.

(1) ([31, Corollary 3.5(c)]) *If a class $\mathcal{M}$ of modules over a ring is closed under pure submodules, then $\mathcal{M}$ is preenveloping if and only if it is closed under direct products.*
(2) ([16, Theorem 2.5]) *If a class $\mathcal{M}$ of modules over a ring is closed under pure quotients, then $\mathcal{M}$ is precovering if and only if it is covering if and only if it is closed under direct sums.*

3. ABSOLUTELY *s*-PURE MODULES

In this section, we give some closure properties of absolutely *s*-pure modules.

Proposition 3.1. *Let B be a right R-module and $A \leq B$. Consider the following statements*:

(1) *A is an s-pure submodule of B*;
(2) *$AI = A \cap BI$ for each maximal left ideal I of R*;
(3) *The map* $\mathrm{Hom}(B, S) \to \mathrm{Hom}(A, S) \to 0$ *is surjective for each simple right R-module S.*

Then (1) $\Leftrightarrow$ (2). *If R is commutative, then these statements are equivalent.*

Proof. (1) $\Leftrightarrow$ (2) By [36, p. 170].

(2) $\Leftrightarrow$ (3) By [12, Proposition 3.1] or [25, Corollary 2.5] □

Remark 3.2. If A is a pure submodule of a right R-module B, then $AI = A \cap BI$ for every left ideal I of R (see, [19, Corollary 4.92]). Therefore, pure submodules are s-pure by Proposition 3.1.

The class of s-pure short exact sequences form a proper class in the sense of Buchsbaum, [8]. This fact gives the following characterization of absolutely s-pure modules (see [26, Proposition 1.12–1.13]).

Lemma 3.3. *The following are equivalent for a right R-module M*:

(1) *M is absolutely s-pure*;
(2) *M is s-pure in $E(M)$*;
(3) *There is an injective module I containing M such that M is s-pure in I*;
(4) *There is an absolutely s-pure module I such that M is s-pure in I*;
(5) *M is s-pure in every extension.*

Now, we give another characterization of absolutely s-pure modules which will be used in the sequel.

Lemma 3.4. *The following are equivalent for a right R-module M*:

(1) *M is absolutely s-pure*;
(2) *For any simple left R-module S, any homomorphism $f: M \to S^+$ factors through an injective right R-module.*

Proof. Let S be a simple left R-module and $f: M \to S^+$ be a homomorphism. Let $E(M)$ be the injective hull of M and $\iota: M \to E(M)$ be the inclusion map. Then the exactness of $0 \to M \otimes S \overset{\iota \otimes 1_S}{\to} E(M) \otimes S$ implies the exactness of $\mathrm{Hom}(E(M), S^+) \overset{\iota_*}{\to} \mathrm{Hom}(M, S^+) \to 0$ and vice versa by [18, Theorem 2.11, Lemma 3.51].

Now assume (1). Then there is a homomorphism $g \in \mathrm{Hom}(E(M), S^+)$ such that $f = g\iota$, and this proves (2).

Assume (2). Then there exist an injective right R-module I, $g : M \to I$, and $h : I \to S^+$ such that $f = hg$. Since I is injective, there is a homomorphism $\alpha : E(M) \to I$ such that $\alpha\iota = g$. So we have the following commutative square:

$$\begin{array}{ccc} M & \xrightarrow{\iota} & E(M) \\ {\scriptstyle f}\downarrow & \searrow^{g} & \downarrow{\scriptstyle \alpha} \\ S^+ & \xleftarrow[h]{} & I \end{array} .$$

Then $f = h\alpha\iota$, and so M_R is absolutely s-pure by Lemma 3.3. □

Remark 3.5. Note that if R is a commutative ring and E an injective cogenerator in $\mathcal{M}od$-R, then $\mathrm{Hom}(S, E) \cong S$. Hence Lemma 3.4 is also hold if we replace S^+ with S.

Proposition 3.6. *The class of absolutely s-pure right R-modules is closed under extensions, direct sums, pure submodules, and direct summands.*

Proof. Absolutely s-pure right R-modules are closed under extensions by [26, Proposition 1.14], and also under direct sums and direct summands by properties of the tensor product. Since every pure exact sequence is s-pure exact, pure submodules of absolutely s-pure modules are absolutely s-pure by Lemma 3.3(4). □

A ring R is called *left SF-ring* if every simple left R-module is flat. A commutative ring R is SF-ring if and only if R is a regular ring. The question, whether a left SF-ring is regular or not is still open. The following is clear by the definitions.

Proposition 3.7. *Every right R-module is absolutely s-pure if and only if R is a left SF-ring.*

4. RINGS CHARACTERIZED BY ABSOLUTELY *s*-PURE AND NEAT-FLAT MODULES

In [1], it is proved that if R is a right N-ring, then a right R-module M is neat-flat if and only if $\mathrm{Tor}_1^R(M, Tr(S)) = 0$ for each simple right R-module S.

Theorem 4.1. *Let R be a right N-ring. Then M is an absolutely s-pure left R-module if and only if* $\mathrm{Ext}_R^1(Tr(S), M) = 0$ *for each simple right R-module S.*

Proof. ($\Rightarrow$) There is an s-pure exact sequence $\mathbb{E} : 0 \to M \to E(M) \xrightarrow{f} K \to 0$ by Lemma 3.3(3). Let S be a simple right R-module. Then the sequence

$$\mathrm{Hom}_R(Tr(S), E(M)) \xrightarrow{f^*} \mathrm{Hom}_R(Tr(S), K) \to \mathrm{Ext}^1(Tr(S), M) \to 0$$

is exact. Since the sequence $\mathbb{E}$ is s-pure exact and R is a right N-ring, f^* is an epimorphism by Theorem 2.1. So that $\mathrm{Ext}_R^1(Tr(S), M) = 0$.

($\Leftarrow$) Consider the exact sequence $\mathbb{E} : 0 \to M \to E(M) \to K \to 0$. Let S be a simple right R-module. Then $\text{Hom}_R(Tr(S), E(M)) \to \text{Hom}_R(Tr(S), K) \to 0$ is exact by the hypothesis, and so $\mathbb{E}$ is s-pure exact by Theorem 2.1. Then M is absolutely s-pure by Lemma 3.3(3). □

Remark 4.2.

(1) By [18, Theorem 9.51], $\text{Tor}_1^R(B^+, A) \cong \text{Ext}_R^1(A, B)^+$ for any finitely presented left R-module B and a left R-module A.
(2) Let R be a right N-ring and M be a right R-module. If K is a pure submodule of M, then $\text{Hom}(S, M) \to \text{Hom}(S, M/K) \to 0$ is an epimorphism for each simple right R-module S by [11, 1.4, p. 12]. Hence K is a neat submodule of M, and in particular flat right R-modules are neat-flat.
(3) By [10, Proof of Proposition 5.3.9.], every right (left) R-module M is a pure submodule of the pure injective right (left) R-module M^{++}.

Proposition 4.3. *Let R be a right N-ring. Then the following statements hold*:

(1) *M is a neat-flat right R-module if and only if M^+ is an absolutely s-pure R-module*;
(2) *M is an absolutely s-pure left R-module if and only if M^+ is a neat-flat R-module*;
(3) *M is an absolutely s-pure left R-module if and only if M^{++} is an absolutely s-pure R-module*;
(4) *M is a neat-flat right R-module if and only if M^{++} is a neat-flat right R-module*;
(5) *The class of absolutely s-pure left R-modules is closed under direct products and pure quotients*;
(6) *The class of neat-flat right R-modules is closed under direct products and pure quotients.*

Proof. (1) This holds by Theorem 4.1, [1, Theorem 4.5] and the standard adjoint isomorphism $\text{Ext}_R^1(Tr(S), M^+) \cong \text{Tor}_1^R(M, Tr(S))^+$.

(2) Let M be a left R-module and S a simple right R-module. Then we have $\text{Tor}_1^R(M^+, Tr(S)) = \text{Ext}_R^1(Tr(S), M)^+$ by Remark 4.2(1). Hence, M is an absolutely s-pure left R-module if and only if M^+ is a neat-flat right R-module by Theorem 4.1 and [1, Theorem 4.5].

(3) and (4) are clear by (1) and (2).

(5) Let $\{M_i\}_{i\in J}$ be a family of absolutely s-pure left R-modules and S be a simple right R-module. Then $\text{Ext}_R^1(Tr(S), \prod_{i\in J} M_i) \cong \prod_{i\in J} \text{Ext}_R^1(Tr(S), M_i) = 0$ by Theorem 4.1. Hence $\prod_{i\in J} M_i$ is absolutely s-pure by Theorem 4.1, again.

Suppose M is an absolutely s-pure left R-module and N a pure submodule of M. Then the exact sequence $0 \to (M/N)^+ \to M^+ \to N^+ \to 0$ splits. By (2), M^+ is neat-flat, and so is $(M/N)^+$. Then M/N is absolutely s-pure by (2), again.

(6) Let $\{M_i\}_{i\in J}$ be a family of neat-flat right R-modules. Then $\bigoplus_{i\in J} M_i$ is neat-flat by [20, Proposition 2.4]. So $(\bigoplus_{i\in J} M_i)^{++} \cong (\prod_{i\in J} M_i^+)^+$ is neat-flat by (4). But $\bigoplus_{i\in J} M_i^+$ is a pure submodule of $\prod_{i\in J} M_i^+$, hence $(\prod_{i\in J} M_i^+)^+ \to (\bigoplus_{i\in J} M_i^+)^+ \to 0$ is a splitting epimorphism. Therefore, $(\bigoplus_{i\in J} M_i^+)^+ \cong \prod_{i\in J} M_i^{++}$ is neat-flat. Since

$\prod_{i\in J} M_i$ is a pure submodule of $\prod_{i\in J} M_i^{++}$, the module $\prod_{i\in J} M_i$ is neat-flat by [1, Theorem 3.2] and [20, Proposition 2.4].

Let N be a pure submodule of a neat-flat right R-module M, then the pure exact sequence $0 \to N \to M \to M/N \to 0$ induces the split exact sequence $0 \to (M/N)^+ \to M^+ \to N^+ \to 0$. Thus $(M/N)^+$ is absolutely s-pure, since M^+ is absolutely s-pure by (1). So M/N is neat-flat by (1), again. □

A right R-module N is said to be *absolutely pure* if it is a pure submodule in every right R-module that contains N, or equivalently, it is pure in its injective hull $E(N)$. Clearly, absolutely pure modules are absolutely s-pure.

Lemma 4.4. *The following are equivalent for a right N-ring R:*

(1) *Every absolutely s-pure left R-module is absolutely pure*;
(2) *Every neat-flat right R-module is flat.*

Proof. (1) ⇒ (2) Let M be a neat-flat right R-module. Then M^+ is absolutely s-pure by Proposition 4.3(1), and so M^+ is absolutely pure by (1). But M^+ is pure injective, so it is injective. Hence M is flat by [18, Theorem 3.52].

(2) ⇒ (1) Let M be an absolutely s-pure left R-module. Then M^+ is neat-flat by Proposition 4.3(2), and so M^+ is flat by (2). Hence M^{++} is injective by [18, Theorem 3.52]. Then M is absolutely pure, since M is a pure submodule of the injective module M^{++}. □

A ring R is said to be a *right C-ring* if every nonzero singular right R-module contains a simple right R-module.

Proposition 4.5. *A left and right Noetherian, and left and right hereditary ring is a left (and right) C-ring.*

Proof. By [23, Proposition 5.4.5], the left R-module R/I has finite length for every essential left ideal I of R. Since R is left Noetherian, R is a left C-ring by [32, Corollary to Theorem 1.2)]. □

A submodule A of a right R-module B is said to be closed in B if there exists no submodule A' of B such that $A \subsetneqq A'$ and A is essential in A'. Over any ring, closed submodules are neat (see [35, Proposition 5]). For a module A, the *singular* submodule $Z(A)$ consists of all elements $a \in A$, such that the annihilator left ideal $(0 : a) = \{r \in R; ra = 0\}$ of which is essential in R. A module A is said to be singular if $Z(A) = A$ and nonsingular if $Z(A) = 0$. If a right R-module M is nonsingular, then for every exact sequence of the form $0 \to K \xrightarrow{f} N \to M \to 0$ we have $f(K)$ is a closed submodule of N by [33, Lemma 2.3]. Hence every nonsingular module is neat-flat.

Theorem 4.6. *Let R be a commutative ring. Consider the following statements:*

(1) *Every neat-flat R-module is flat and R is nonsingular*;
(2) *R is semihereditary.*

Then (1) ⇒ (2). *If R is Noetherian, then* (2) ⇒ (1).

Proof. (1) $\Rightarrow$ (2) Let M be a nonsingular module. As noted above, M is neat-flat, and so M is flat by (1). Then R is semihereditary by [15, Proposition 2.3].

(2) $\Rightarrow$ (1) First note that R is a C-ring by Proposition 4.5. Let M be a neat-flat R-module. Consider the sequence $0 \to H \to F \to M \to 0$ with F projective. Then H is neat in F by [1, Lemma 3.1], and it is closed in F by [14, Theorem 5]. Since R is semihereditary, R is nonsingular. So that F is nonsingular. Then M is nonsingular by [33, Lemma 2.3]. Now, M is flat by [15, Proposition 2.3]. □

By [10, p. 132], a ring R is right noetherian if and only if every absolutely pure right R-module is injective. For absolutely s-pure modules, we have the following result.

Theorem 4.7. *The following are equivalent for a commutative ring R:*

(1) *Every absolutely s-pure R-module is injective and R is nonsingular*;
(2) *R is a (semi)hereditary Noetherian ring.*

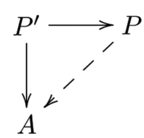
Proof. (1) $\Rightarrow$ (2) Note that each absolutely pure R-module is absolutely s-pure, and so injective by (1). Then R is Noetherian by [10, p. 132]. The rest of (2) follows by Lemma 4.4 and Theorem 4.6.

(2) $\Rightarrow$ (1) By Lemma 4.4 and Theorem 4.6, absolutely s-pure R-modules are absolutely pure. But R is noetherian, so every absolutely s-pure R-module is injective by [10, p. 132]. □

Corollary 4.8. *The following are equivalent for a commutative domain R:*

(1) *Every absolutely s-pure module is injective*;
(2) *R is a Dedekind domain.*

Following Megibben [24], an R-module A is absolutely pure if and only if every diagram



with P' finitely generated and P projective can be completed to a commutative diagram.

Recall that, a ring R is *right IF ring* if every injective right R-module is flat. It is known that, R is a right *IF* ring if and only if every finitely presented right R-module is a submodule of a free module ([3, Theorem 1]). A ring R is called *right Kasch* if any simple right R-module embeds in R. Now, we consider when every injective right R-module is neat-flat.

By [1, Theorem 3.2] a right R-module M is neat-flat if and only if it is simple-projective.

Theorem 4.9. *For any ring R, the following conditions are equivalent*:

(1) *R is a right Kasch ring*;
(2) *Every absolutely pure right R-module is neat-flat*;
(3) *Every injective right R-module is neat-flat*;
(4) *The injective hull of every simple right R-module is neat-flat.*

Proof. (1) $\Rightarrow$ (2) Let E be an absolutely pure right R-module and M a simple right R-module. Since R is a right Kasch ring, there is an embedding $\iota : M \to R$. Let $f : M \to E$ be a homomorphism. As E is absolutely pure, there is a homomorphism $g : R \to E$ such that $f = g\iota$. That is, E is simple-projective. Hence E is neat-flat by [1, Theorem 3.2].

(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are trivial.

(4) $\Rightarrow$ (1) Let S be a simple right R-module and $u : S \to E(S)$ be the inclusion homomorphism. Since $E(S)$ is neat-flat, by [1, Theorem 3.2], there is a finitely generated free module F and homomorphisms v, w such that the following diagram commutes:

$$\begin{array}{ccccc} 0 & \longrightarrow & S & \stackrel{u}{\longrightarrow} & E(S) \\ & & \downarrow{\scriptstyle v} & \nearrow{\scriptstyle w} & \\ & & F & & \end{array} .$$

Since wv is a monomorphism, v is a monomorphism, and so (1) holds. □

Corollary 4.10. *Let R be a ring. Then R is a right Kasch ring if and only if for every free left R-module F, F^+ is neat-flat.*

Proof. Suppose R is a right Kasch ring, and let F be a free left R-module. By [18, Theorem 3.52], F^+ is an injective right R-module. Then F^+ is neat-flat by Theorem 4.9. Conversely, let M be any injective right R-module. There is a free left R-module F and an epimorphism $F \to M^+$ from which we obtain an exact sequence $0 \to M^{++} \to F^+$. Since F^+ is neat-flat and $M \leq M^{++}$, M is a direct summand of F^+, and so M is neat-flat. Hence R is a right Kasch ring by Theorem 4.9. □

Corollary 4.11. *If R is a right coherent and right Kasch ring, then $_RR$ is absolutely s-pure.*

Proof. Every simple right R-module S embeds in R, hence S finitely presented. Thus R is a right N-ring. Since R is right Kasch, $_RR^+$ is neat-flat by Corollary 4.10. Then $_RR$ is absolutely s-pure by Proposition 4.3. □

A ring R is called right $\sum$-CS if closed submodules of projective right R-modules are direct summand. $\sum$-CS rings were first introduced by Oshiro [27] under the name co-H-rings. By [1, Theorem 3.5], R is a right $\sum$-CS ring if and only if every neat-flat right R-module is projective. R is a QF ring if and only if every injective right R-module is projective, if and only if every projective right R-module is injective. Oshiro [27] proved that R is a QF ring if and only if R is a right $\sum$-CS ring and $Z(R_R) = J(R)$.

Corollary 4.12. *A ring R is right Kasch and right $\sum$-CS if and only if R is QF.*

Proof. Necessity: By [1, Theorem 3.5], every neat-flat right R-module is projective. Thus every injective right R-module is projective by Theorem 4.9, hence R is QF. Conversely, R is right Kasch by Theorem 4.9, and R is right $\sum$-CS by [27, Theorem 4.3]. □

Jensen [17, Proposition 1.4] proved that if R is a left coherent ring, then every pure injective flat right R-module is projective if and only if R is right perfect.

We obtain the following characterization of right $\sum$-CS rings.

Theorem 4.13. *Let R be a right N-ring. Then the following are equivalent*:

(1) *R is a right $\sum$-CS ring*;
(2) *Every pure injective neat-flat right R-module is projective*;
(3) *For any left R-module M, M is absolutely s-pure if and only if M^+ is projective*;
(4) *Every absolutely s-pure left R-module is injective and R is right perfect.*

Proof. (1) $\Rightarrow$ (2) Follows by [1, Theorem 3.5].

(2) $\Rightarrow$ (3) Let M be a left R-module. By Proposition 4.3, M is absolutely s-pure if and only if M^+ is neat-flat. Since M^+ is pure injective, (2) completes the proof of (3).

(3) $\Rightarrow$ (1) Firstly, we show that R is left coherent and right perfect. Let F be an absolutely s-pure left R-module. Then F^{++} is injective by [18, Theorem 3.52], since F^+ is projective by (3). Since the monomorphism $F \to F^{++}$ is pure and F^{++} is injective, F is absolutely pure. Then, the classes of absolutely s-pure R-modules and absolutely pure R-modules coincide. Hence F is absolutely pure if and only if F^+ is projective by (3). Then R is left coherent and right perfect by [2, Theorem 3].

Let M be a neat-flat right R-module. We claim that M is a flat right R-module. By Proposition 4.3 and (3), M^{++} is projective. Consider the pure exact sequence

$$0 \to M \to M^{++} \to M^{++}/M \to 0.$$

Since flat modules are closed under pure submodules, M is flat. By the first part of the proof M is projective, since R is right perfect. Then R is a right $\sum$-CS ring by [1, Theorem 3.5].

(3) $\Rightarrow$ (4) In the proof of (3) $\Rightarrow$ (1), we show that the classes of absolutely s-pure R-modules and absolutely pure R-modules coincide. Since the condition (3) implies R is a right $\sum$-CS ring, R is left Artinian by [28, Proposition 3.2]. Hence R is right perfect, and every absolutely s-pure left R-module is injective by [2, Theorem 2].

(4) $\Rightarrow$ (1) The condition (4) implies that every neat-flat right R-module is flat by Lemma 4.4. But R is right perfect, so neat-flat right R-modules are projective. Hence R is right $\sum$-CS by [1, Theorem 3.5]. □

5. COVERS AND ENVELOPES

In this section, we use the results in previous sections to study the enveloping and covering properties of absolutely s-pure modules and neat-flat modules.

Proposition 5.1. *Let R be a right N-ring. The following statements hold*:

(1) *Every left R-module has an absolutely s-pure preenvelope*;
(2) *Every left R-module has an absolutely s-pure cover*;
(3) *Every right R-module has a neat-flat preenvelope*;
(4) *Every right R-module has a neat-flat cover.*

Proof. (1) Absolutely s-pure left R-modules are closed under pure submodules by Proposition 3.6. Then the claim follows by Proposition 4.3(5) and Lemma 2.2(1).

(2) Absolutely s-pure left R-modules are closed under direct sums and pure quotients by Proposition 3.6 and Proposition 4.3(5). Hence every R-module has an absolutely s-pure cover by Lemma 2.2(2).

(3) Neat-flat right R-modules are closed under direct product by Proposition 4.3(6) and pure submodules by [20, Proposition 2.4]. Then (3) follows by Lemma 2.2(1).

(4) Neat-flat right R-modules are closed under pure quotients by Proposition 4.3(6), and under direct sums by [1, Proposition 3.4.]. Hence every module has a neat-flat cover by Lemma 2.2(2). □

A left R-module E is called *s-pure injective* if it is injective with respect to s-pure short exact sequences. Note that for each simple right R-module S, S^+ is an s-pure injective left R-module by the standard adjoint isomorphism.

Proposition 5.2. *Absolutely s-pure cover of an s-pure injective left R-module is injective.*

Proof. Let M be an s-pure injective left R-module. Let $f : F \to M$ be an absolutely s-pure cover of M. By Lemma 3.3, there is an s-pure exact sequence $0 \to F \xrightarrow{i} E \to L \to 0$ with E injective. Since M is s-pure injective, there exists a homomorphism $g : E \to M$ such that $f = gi$. Since E is absolutely s-pure, there exists $\alpha : E \to F$ such that $g = f\alpha$. Therefore $f = gi = f\alpha i$, and so $\alpha i = 1_F$. It follows that, F is isomorphic to a direct summand of E, and hence F is injective. □

Recall that every left R-module has an epic flat envelope if and only if R is a right semihereditary ring, [31, Corolary 4.3]. It is well known that, R is a right semihereditary ring if and only if every right R-module has a monic absolutely pure cover if and only if every homomorphic image of an injective right R-module is absolutely pure, (see, [31, Corolary 4.13] and [7, Corollary 3.8]). Next, we consider when every left R-module has a monic absolutely s-pure cover.

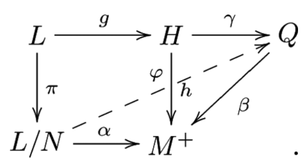
Theorem 5.3. *The following are equivalent for a ring R:*

(1) *Every s-pure injective left R-module has a monic injective cover;*
(2) *Every quotient of any absolutely s-pure left R-module is absolutely s-pure;*
(3) *Every quotient of any injective left R-module is absolutely s-pure;*
(4) *Every left R-module has a monic absolutely s-pure cover.*

When R is a right N-ring, these conditions are equivalent to the following ones:

(5) *Every submodule of any neat-flat right R-module is neat-flat;*
(6) *Every simple right R-module has an epic projective envelope;*
(7) *Every right R-module has an epic neat-flat envelope;*
(8) *For every simple right R-module S either $S^* = 0$ or S is projective (i.e., R is a right PS ring).*

Proof. (1) $\Rightarrow$ (2) Let L be an absolutely s-pure left R-module and $N \leq L$. Let M be a simple right R-module. For any homomorphism $\alpha : L/N \to M^+$, there exists an injective left R-module H, $g : L \to H$ and $h : H \to M^+$ such that $\alpha\pi = hg$ by Lemma 3.4, where $\pi : L \to L/N$ is the canonical epimorphism. By (1), M^+ has a monic injective cover $\beta : Q \to M^+$. Thus there exists $\gamma : H \to Q$ such that $h = \beta\gamma$, which implies that $\mathrm{Im}(\alpha) \subseteq \mathrm{Im}(\beta)$ and so there exists $\varphi : L/N \to Q$ such that $\beta\varphi = \alpha$.



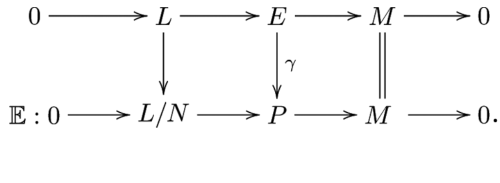
That is, α factors trough the injective module Q. Therefore, L/N is absolutely s-pure by Lemma 3.4.

(2) $\Rightarrow$ (4) Every pure quotient of any absolutely s-pure left R-module is absolutely s-pure by (2). By Proposition 3.6, absolutely s-pure left R-modules are also closed under direct sums. Now, the claim follows by [13, Proposition 4].

(4) $\Rightarrow$ (1) by Proposition 5.2.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (2) Suppose that N is a submodule of an absolutely s-pure left R-module L. Then there is an s-pure exact sequence $0 \to L \to E \to M \to 0$ with E injective by Lemma 3.3. We have the pushout diagram



Since s-pure exact sequences are closed under pushout, $\mathbb{E}$ is also s-pure exact. On the other hand, γ is an epimorphism, and so P is absolutely s-pure by (3). Therefore, L/N is absolutely s-pure by Lemma 3.3.

(2) $\Rightarrow$ (5) Suppose N is a submodule of a neat-flat right R-module L. Then L^+ absolutely s-pure by Proposition 4.3. Clearly, N^+ is an epimorphic image of L^+, and so N^+ is absolutely s-pure by (2). Hence N is neat-flat by Proposition 4.3, again.

(5) $\Rightarrow$ (2) Suppose that N is a submodule of an absolutely s-pure left R-module L. We claim that L/N is absolutely s-pure. We have an exact sequence $0 \to (L/N)^+ \to L^+ \to N^+ \to 0$ where L^+ neat-flat, since L is absolutely s-pure by Proposition 4.3. Then $(L/N)^+$ is neat-flat by (5), and so L/N absolutely s-pure by Proposition 4.3, again.

(5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) $\Leftrightarrow$ (8) By [1, Theorem 3.2] and [20, Theorem 3.7]. □

Theorem 5.4. *Let R be a ring. Consider the following statements*:

(1) *Every left R-module has a monic absolutely s-pure cover*;
(2) *Every simple left R-module has a monic injective cover*;
(3) *A simple left R-module S is either injective or* $\mathrm{Hom}(E, S) = 0$ *for each injective left R-module E.*

Then (2) $\Leftrightarrow$ (3). *If R is commutative, then all these statements are equivalent.*

Proof. First note that if R is commutative, then every simple R-module is s-pure injective by Proposition 3.1(3).

(1) $\Rightarrow$ (2) Since simple modules are s-pure injective, (2) follows by Theorem 5.3.

(2) $\Rightarrow$ (1) Similar to that proof of (1) $\Rightarrow$ (2) in Theorem 5.3, one can show that quotients of absolutely s-pure modules are absolutely s-pure. So, the claim follows by Theorem 5.3.

(2) $\Rightarrow$ (3) Let S be a simple left R-module. Suppose S is not injective. Then S has a monic injective cover $f : Q \to S$ by (2). Since S is simple and f is monic, $Q = 0$. Now, let E be an injective left R-module and $h \in \mathrm{Hom}(E, S)$. Then there is a homomorphism $g : E \to Q$ such that $h = fg = 0$. This proves (3).

(3) $\Rightarrow$ (2) Let S be a simple R-module. Then, by (3), S is either injective or $\mathrm{Hom}(E, S) = 0$ for each injective module E. If S is injective, then $1_S : S \to S$ is a monic injective cover of S. If $\mathrm{Hom}(E, S) = 0$ for each injective module E, then $0 \to S$ is a monic injective cover of S. □

Remark 5.5.

(1) For a left small ring R, i.e., $\mathrm{Rad}(E) = E$ for every injective left R-module E, we have $\mathrm{Hom}(E, S) = 0$ for each simple left R-module S. If R is a left V-ring then every simple left R-module is injective. Hence, each simple left R-module has a monic injective cover over left small rings and over left V-rings.

(2) Let R be a commutative semihereditary ring and S a simple R-module. Suppose $\mathrm{Hom}(E, S) \neq 0$ for some injective R-module E. Then $S \cong E/K$ for some $K \leq E$, and so S is absolutely pure by [7, Corollary 3.8]. But S is also s-pure injective by Proposition 3.1(3), so it is injective. Thus every simple R-module has a monic injective cover by Theorem 5.4(3).

For a left coherent ring R, Mao and Ding [22] proved that, ${}_RR$ is absolutely pure if and only if every (finitely presented) left R-module has an epic absolutely pure cover [22, Corollary 3.2].

Theorem 5.6. *Let R be a right N-ring. Then the following statements are equivalent*:

(1) *R is a right Kasch ring*;
(2) *Every left R-module has an epic absolutely s-pure cover*;
(3) *Every flat left R-module is absolutely s-pure*;
(4) *R is left absolutely s-pure.*

Proof. (4) ⇒ (1) If ${}_RR$ is absolutely s-pure, then every free R-module F is absolutely s-pure. By Proposition 4.3, F^+ is neat-flat. Hence R is right Kasch by Corollary 4.10.

(1) ⇒ (2) Since R is a right N-ring, every left R-module has an absolutely s-pure cover by Proposition 5.1. As R is a right Kasch ring, ${}_RR$ is absolutely s-pure by (4) ⇔ (1). Hence any absolutely s-pure cover is epic by [10, p. 106].

(2) ⇒ (3) Let F be a flat left R-module and $\varphi : M \to F$ be an epic absolutely s-pure cover of F. Then the pure exact sequence $0 \to \mathrm{Ker}(\varphi) \to M \to F \to 0$ induces the splitting exact sequence $0 \to F^+ \to M^+ \to (\mathrm{Ker}(\varphi))^+ \to 0$. Thus F^+ is neat-flat, since M^+ is neat-flat by Proposition 4.3. So F is absolutely s-pure by Proposition 4.3.

(3) ⇒ (4) is trivial. □

We conclude the paper with the following remark.

Remark 5.7. Let R be a commutative ring. By Proposition 3.1(3), every simple module is s-pure injective. Hence absolutely s-pure cover of a simple module is injective by Proposition 5.2. Actually, by using the same method in the proof of Proposition 5.2 one can easily shows that, absolutely pure cover of an s-pure injective left R-module is injective. Note that, if R is left coherent, then every left R-module has an absolutely pure cover by [30, Corollary 2.7]. Hence, if R is a coherent ring, then every simple R-module has an injective cover. If R is coherent and absolutely pure, then every simple module has an epic injective cover by [22, Corollary 3.2].

ACKNOWLEDGMENTS

Some part of this paper was written while the second author was visiting Padova University, Italy. He wishes to thank the members of the Department of Mathematics for their kind hospitality.

FUNDING

The second author thanks the Scientific and Technical Research Council of Turkey (TÜBİTAK) for their financial support.

REFERENCES

[1] Büyükaşık, E., Durğun, Y. *Neat-flat modules*. Available at http://arxiv.org/abs/1306.2860. Accessed June 26, 2014.
[2] Cheatham, T. J., Stone, D. R. (1981). Flat and projective character modules. *Proc. Amer. Math. Soc.* 81:175–177.
[3] Colby, R. R. (1975). Rings which have flat injective modules. *J. Algebra* 35:239–252.
[4] Crivei, I. (2005). *s*-pure submodules. *Int. J. Math. Math. Sci.* 4:491–497.
[5] Crivei, S. (1998). *m*-injective modules. *Mathematica*, 40(63):71–78.
[6] Crivei, S. (2013). Neat and coneat submodules of modules over commutative rings. *Bull. Aust. Math. Soc.*
[7] Crivei, S., Torrecillas, B. (2008). On some monic covers and epic envelopes. *Arab. J. Sci. Eng. Sect. C Theme Issues* 33:123–135.
[8] Buchsbaum, D. A. (1959). A note on homology in categories. *Ann. of Math. (2)* 69:66–74.
[9] Dinh, H. Q., Holston, C. J., Huynh, D. V. (2013). Some results on V-rings and weakly V-rings. *J. Pure Appl. Algebra* 217:125–131.
[10] Enochs, E. E., Jenda, O. M. G. (2000). *Relative Homological Algebra*. de Gruyter Expositions in Mathematics, Vol. 30. Berlin: Walter de Gruyter & Co.
[11] Facchini, A. (1998). *Module Theory*. Progress in Mathematics, Vol. 167. Basel: Birkhäuser Verlag. Endomorphism rings and direct sum decompositions in some classes of modules.
[12] Fuchs, L. (2012). Neat submodules over integral domains. *Period. Math. Hungar.* 64:131–143.
[13] García Rozas, J. R., Torrecillas, B. (1994). Relative injective covers. *Comm. Algebra* 22:2925–2940.
[14] Generalov, A. I. (1978). Weak and ω-high purities in the category of modules. *Mat. Sb. (N.S.)* 105(147):389–402, 463.
[15] Goodearl, K. R. (1972). *Singular Torsion and the Splitting Properties*. Providence, R. I.: American Mathematical Society. Memoirs of the American Mathematical Society, No. 124.
[16] Holm, H., Jørgensen, P. (2008). Covers, precovers, and purity. *Illinois J. Math.* 52:691–703.
[17] Jensen, C. U., Simson, D. (1979). Purity and generalized chain conditions. *J. Pure Appl. Algebra* 14:297–305.
[18] Rotman, J. (1979). *An Introduction to Homological Algebra*. New York: Academic Press.
[19] Lam, T. Y. (1999). *Lectures on Modules and Rings*. Graduate Texts in Mathematics, Vol. 189. New York: Springer-Verlag.
[20] Mao, L. (2007). When does every simple module have a projective envelope? *Comm. Algebra* 35:1505–1516.
[21] Mao, L. (2011). Modules characterized by their simple submodules. *Taiwanese J. Math.* 15:2337–2349.
[22] Mao, L., Ding, N. (2007). *FI*-injective and *FI*-flat modules. *J. Algebra* 309:367–385.
[23] McConnell, J. C., Robson, J. C. (2001). *Noncommutative Noetherian Rings*. Revised, Graduate Studies in Mathematics, Vol. 30. Providence, RI: American Mathematical Society. With the cooperation of L. W. Small.
[24] Megibben, C. (1970). Absolutely pure modules. *Proc. Amer. Math. Soc.* 26:561–566.

[25] Mermut, E., Santa-Clara, C., Smith, P. F. (2009). Injectivity relative to closed submodules. *J. Algebra* 321:548–557.

[26] Mišina, A. P., Skornjakov, L. A. (1976). *Abelian Groups and Modules, Algebra, Logic and Applications*. Amer. Math. Soc. Transl. Ser. 2, Vol. 107, Providence, RI.

[27] Oshiro, K. (1984). Lifting modules, extending modules and their applications to QF-rings. *Hokkaido Math. J.* 13:310–338.

[28] Oshiro, K. (1989). On Harada rings. I, II. *Math. J. Okayama Univ.* 31:161–178, 179–188.

[29] Parra, R., Rada, J. (2011). Projective envelopes of finitely generated modules. *Algebra Colloq.* 18:801–806.

[30] Pinzon, K. (2008). Absolutely pure covers. *Comm. Algebra* 36:2186–2194.

[31] Rada, J., Saorin, M. (1998). Rings characterized by (pre)envelopes and (pre)covers of their modules. *Comm. Algebra* 26:899–912.

[32] Renault, G. (1964). Étude de certains anneaux a liés aux sous-modules compléments dun a-module. *C. R. Acad. Sci. Paris* 259:4203–4205.

[33] Sandomierski, F. L. (1968). Nonsingular rings. *Proc. Amer. Math. Soc.* 19:225–230.

[34] Sklyarenko, E. G. (1978). Relative homological algebra in categories of modules. *Russian Math. Surveys* 33:97–137. Traslated from Russian from *Uspehi Mat. Nauk* 33(201):85–120.

[35] Stenström, B. T. (1967). High submodules and purity. *Ark. Mat.* 7:173–176.

[36] Stenström, B. T. (1967). Pure submodules. *Ark. Mat.* 7:159–171.

[37] Wang, M. Y., Zhao, G. (2005). On maximal injectivity. *Acta Math. Sin. (Engl. Ser.)* 21:1451–1458.

[38] Xiang, Y. (2010). Max-injective, max-flat modules and max-coherent rings. *Bull. Korean Math. Soc.* 47:611–622.

[39] Xu, J. (1996). *Flat Covers of Modules*. Lecture Notes in Mathematics, Vol. 1634. Berlin: Springer-Verlag.